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HOMOTOPY OPERATIONS FOR SIMPLICIAL COMMUTATIVE ALGEBRAS[1]

BY

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ABSTRACT. The indicated operation algebra is studied by methods dual to the usual ones for studying the Steenrod algebra. In particular, the operations are constructed using higher symmetries of the shuffle map and their "Adem relations" are computed using the transfer map in the cohomology of symmetric groups.

1. Introduction. There is a general theorem to the effect that if **C** is a reasonable category, for instance a category of universal algebras, then the category of simplicial objects over **C** has a *homotopy theory* in Quillen's sense [**10**, II, §4]. If **C** is the category of groups this homotopy theory is equivalent to the usual geometric homotopy theory of spaces, but there are a lot of other interesting choices for **C** [**12**], [**11**], [**9**]. The purpose of this paper is to calculate the "ring" of unstable homotopy operations in the homotopy theory that results from taking **C** to be the category of commutative (associative) $\mathbf{Z}/2$-algebras. The motivation for doing this will become clear in subsequent papers; it turns out that this "ring" operates on

(a) most second-quadrant mod 2 cohomology spectral sequences, and on

(b) most first-quadrant mod 2 homology spectral sequences of infinite loop spaces

in a way that intertwines closely with the respective actions of the Steenrod algebra and the Dyer-Lashof algebra.

There is a completely different approach to many of the results of this paper, in particular to Theorem 2.1, in unpublished notes of A. K. Bousfield [**13**]. However the machinery developed below may be interesting in its own right, and some of it is needed for the applications we have in mind.

2. Outline of results. Suppose that V is a simplicial commutative $\mathbf{Z}/2$-algebra. In particular then V is a simplicial abelian group; let NV denote the normalized chain complex of V [7, p. 236] and define $\pi_k V$ ($k \geqslant 0$), the kth *homotopy group* of V, by

$$\pi_k V = H_k(NV), \qquad [\mathbf{8}, \text{p. } 94].$$

According to the Eilenberg-Zilber theorem [7, p. 238], the multiplication map of V induces maps

$$\pi_i V \otimes \pi_j V \to \pi_{i+j} V \qquad (i, j \geqslant 0)$$

which give $\pi_* V$ the structure of a graded commutative $\mathbf{Z}/2$-algebra.

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2.1. THEOREM. *Suppose that V is a simplicial commutative $\mathbf{Z}/2$-algebra and that $x \in \pi_n V$. Then there exist naturally associated elements $\delta_i x \in \pi_{n+i} V$ $(2 \leqslant i \leqslant n)$ with the following properties*:

(i) *if* $y \in \pi_n V$, *then*

$$\delta_i(x+y) = \delta_i x + \delta_i y, \qquad 2 \leqslant i < n, \quad \textit{and}$$
$$\delta_n(x+y) = \delta_n x + \delta_n y + xy,$$

(ii) *if* $y \in \pi_m V$ *and* $2 \leqslant k \leqslant m+n$, *then*

$$\delta_k(xy) = \begin{cases} y^2\delta_k(x) & \textit{if } m = 0, \\ x^2\delta_k(y) & \textit{if } n = 0, \\ 0 & \textit{if } m > 0 \textit{ and } n > 0, \end{cases}$$

(iii) *if* $j < 2i$, *then*

$$\delta_j\delta_i(x) = \sum_{(j+1)/2 \leqslant k \leqslant i-1} \binom{i-j+k-1}{i-k} \delta_{i+j-k}\delta_k(x).$$

2.2. REMARK. Let $\delta_0\colon \pi_n V \to \pi_n V$ be the identity map. The techniques of §§5–6 show that by iterating the operations δ_i $(i \neq 1)$ and forming products it is possible to construct all homotopy operations on the category of simplicial commutative $\mathbf{Z}/2$-algebras.

2.3. RELATIONSHIP TO $H_*(K(\mathbf{Z}, n); \mathbf{Z}/2)$. Let $\Delta[n]$ be the standard n-simplex [8, p. 14], $\mathring{\Delta}[n]$ its boundary, and let S^n $(n > 0)$ denote the simplicial $\mathbf{Z}/2$-module which is the quotient of the free $\mathbf{Z}/2$-module on $\Delta[n]$ by the free $\mathbf{Z}/2$-module on $\mathring{\Delta}[n]$. Note that

$$\pi_i S^n = H_i(\Delta[n], \mathring{\Delta}[n]; \mathbf{Z}/2) = \begin{cases} 0, & i \neq n, \\ \mathbf{Z}/2, & i = n, \end{cases}$$

where, as above, $\pi_* S^n = H_*(NS^n)$. Let $\operatorname{Sym} S^n = \bigoplus_{k \geqslant 0} \operatorname{Sym}_k S^n$ be the result of applying dimensionwise to S^n the symmetric algebra functor.

Suppose that V is a simplicial commutative $\mathbf{Z}/2$-algebra, and let "[–]" denote "simplicial homotopy classes" of the indicated type of map. Then

$$\begin{aligned} \pi_n(V) &= \left[\text{pointed simplicial set maps } \Delta[n]/\mathring{\Delta}[n] \to V\right] \\ &= \left[\text{simplicial } \mathbf{Z}/2\text{-module maps } S^n \to V\right] \\ &= \left[\text{simplicial algebra maps Sym } S^n \to V\right] \end{aligned}$$

where the first equality comes from [8, pp. 94, 15] and the second two from adjointness arguments. It follows that natural transformations $\pi_n(-) \dot{\to} \pi_m(-)$ on the category of simplicial commutative $\mathbf{Z}/2$-algebras are in 1-1 correspondence with $\pi_m(\operatorname{Sym} S^n)$. By the Dold-Thom theorem and the Steenrod splitting theorem [5] there are natural isomorphisms

$$\begin{aligned} \pi_* \operatorname{Sym}(S^n) &= H_*\big(SP^\infty(\Delta[n]/\mathring{\Delta}[n]); \mathbf{Z}/2\big) \\ &= H_*(K(Z, n); \mathbf{Z}/2), \qquad n > 0, \end{aligned}$$

where $SP^\infty(-)$ is the reduced infinite symmetric product functor. From this point of view the present paper is taken up with constructing certain natural classes in the mod 2 homology of $K(Z, n)$'s and studying their behavior under a particular (nontopological) form of composition. These classes are actually higher divided squares and already appear in the work of Cartan [2].

2.4. ORGANIZATION OF THE PAPER. §3 uses an acyclic models argument to construct higher Eilenberg-Mac Lane maps. In §4 these maps are used to define the operations δ_i and prove 2.1(i). §5 develops a relationship between the homotopy of symmetric powers and the cohomology of symmetric groups; this is used in §6 to prove 2.1(ii)–(iii).

3. Higher Eilenberg-Mac Lane maps. In this section we will exploit the symmetry of the Eilenberg-Mac Lane shuffle map in what may seem a slightly peculiar way. In fact, this is the reverse of the way in which the assymmetry of the Alexander-Whitney map is exploited in the chain level construction of Steenrod operations [5].

If V and W are simplicial $\mathbf{Z}/2$-modules, let $V \otimes W$ denote their dimensionwise tensor product

$$(V \otimes W)_n = V_n \otimes W_n$$

with the tensor product face and degeneracy operators. If A and B are graded $\mathbf{Z}/2$-modules, let $A \hat{\otimes} B$ denote their graded tensor product

$$(A \hat{\otimes} B)_n = \bigoplus_{i+j=n} (A_i \otimes B_j).$$

The Leibnitz differential

$$\partial(a \hat{\otimes} b) = (\partial a) \hat{\otimes} b + a \hat{\otimes} \partial b$$

makes $A \hat{\otimes} B$ into a chain complex if A and B are chain complexes.

Let $C(V)$ denote the unnormalized chain complex [7, p. 235] of the simplicial $\mathbf{Z}/2$-module V. Note that the chain complexes $N(V \otimes W)$ and $C(V) \hat{\otimes} C(W)$ both have $V_0 \otimes W_0$ in dimension zero.

3.1. PROPOSITION (EILENBERG-ZILBER). *Let V and W be simplicial $\mathbf{Z}/2$-modules. Then there is one and only one natural chain map*

$$D: C(V) \hat{\otimes} C(W) \to N(V \otimes W)$$

which is the identity map in dimension 0. *Moreover, this chain map is a chain homotopy equivalence.*

This will be proved below.

3.2. REMARK. There is an analogous proposition for tensor products with more than one factor. A naturality argument can be used to show that the map D of 3.1 factors through the surjective chain homotopy equivalence [7, p. 236] $C(V) \hat{\otimes} C(W) \to N(V) \hat{\otimes} N(W)$. This also follows computationally from the fact that D is necessarily the shuffle map [7, p. 243].

For $n \geq 0$, let

$$\phi_n: C(V) \hat{\otimes} C(W) \to N(V \otimes W)$$

be the degree $(-n)$ map of graded $\mathbf{Z}/2$-modules given by

$$\phi_n(v \hat{\otimes} w) = \begin{cases} 0, & \text{if } \deg v \neq n \text{ or } \deg w \neq n, \\ \text{residue class of } v \otimes w \text{ in } N(V \otimes W)_n, & \text{otherwise.} \end{cases}$$

Let T denote the switching map

$$C(V) \hat{\otimes} C(W) \to C(W) \hat{\otimes} C(V) \quad \text{or} \quad N(V \otimes W) \to N(W \otimes V).$$

3.3. PROPOSITION. *Let D be as in* 3.1, *and let V and W be simplicial $\mathbf{Z}/2$-modules. Then there exist natural maps*

$$D^k\colon (C(V) \hat{\otimes} C(W))_{i+k} \to N(V \otimes W)_i \qquad (0 \leqslant k \leqslant i)$$

such that

(i) $D^0 + TD^0T = D + \phi_0$ *and*

(ii) $D^k + TD^kT = D^{k-1}\partial + \partial D^{k-1} + \phi_k$ $(k > 0)$.

Moreover, if $\{\Delta^k\}$ is another such collection of maps, then there exist natural maps

$$E^k\colon (C(V) \hat{\otimes} C(W))_{i+k} \to N(V \otimes W)_i \qquad (0 \leqslant k \leqslant i)$$

such that

(iii) $E^0 + TE^0T = D^0 + \Delta^0$, *and*

(iv) $E^k + TE^kT = D^k + \Delta^k + E^{k-1}\partial + \partial E^{k-1}$ $(k > 0)$.

REMARK. Equations (ii) and (iv) above only make sense when applied to an element in $C(V) \hat{\otimes} C(W)$ of degree $\geqslant 2k$; the maps D^k and E^k are not defined on elements of degree $< 2k$.

PROOF OF 3.1. Let $X[i]$ be the free $\mathbf{Z}/2$-module on $\Delta[i]$ and let $\Delta_i \in X[i]_i$ be the image of the universal i-simplex [8, p. 14]. If V is any simplicial $\mathbf{Z}/2$-module, the map $f \mapsto f(\Delta_i)$ establishes a natural isomorphism

$$V_i \approx \operatorname{Hom}(X[i], V).$$

It follows from a standard representability argument [7, p. 240] that to give a natural map

$$D\colon C(V) \hat{\otimes} C(W) \to N(V \otimes W)$$

it is necessary and sufficient to specify the images ξ_{ij} in $N(X[i] \otimes X[j])_{i+j}$ of $\Delta_i \hat{\otimes} \Delta_j$ for all $i, j \geqslant 0$.

We will find the ξ_{ij}'s by induction on $i + j$, and simultaneously show that they are unique. The choice $\xi_{00} = \Delta_0 \otimes \Delta_0$ is forced by the requirement that D_0 be the identity map.

Suppose that ξ_{ij} have been chosen for $i + j < n$ in such a way that the corresponding natural maps

$$D_k\colon (C(V) \hat{\otimes} C(W))_k \to N(V \otimes W)_k \qquad (k < n)$$

satisfy

$$\partial D_k = D_{k-1}\partial, \qquad 1 \leqslant k < n.$$

It is necessary to find elements ξ_{ij} in $N(X[i] \otimes X[j])_n$, $i + j = n$, such that

$$\partial \xi_{ij} = D_{n-1}(\partial(\Delta_i \hat{\otimes} \Delta_j)).$$

By the inductive hypothesis,

$$\partial D_{n-1}(\partial(\Delta_i \hat{\otimes} \Delta_j)) = D_{n-2}\partial^2(\Delta_i \hat{\otimes} \Delta_j) = 0$$

so, since $N(X[i] \otimes X[j])$ is acyclic in positive dimensions, it is certainly possible to find elements ξ_{ij} $(i + j = n)$ with the correct boundaries. However, $N(X[i] \otimes X[j])$ vanishes above dimension $i + j$ (because for simple combinatorial reasons the product $\Delta[i] \times \Delta[j]$ has no nondegenerate simplices above dimension $i + j$) so, again by acyclicity,

$$\partial\colon N(X[i] \otimes X[j])_{i+j} \to N(X[i] \otimes X[j])_{i+j-1}$$

is injective. This shows that there is only one choice for ξ_{ij}.

The fact that D is a chain homotopy equivalence follows from the standard version of the Eilenberg-Zilber theorem [7, VIII, 8].

Proof of 3.3. The maps D^k are constructed by induction on k. As in the proof of 3.1 it suffices to specify elements $\xi_{ij}^k = D^k(\Delta_i \hat{\otimes} \Delta_j)$ in $N(X[i] \otimes X[j])_{i+j-k}$ for $i + j \geqslant 2k$.

The element ξ_{00}^0 is necessarily zero. The elements ξ_{ij}^0 for $i + j > 0$ must satisfy

$$\xi_{ij}^0 + T\xi_{ji}^0 = D(\Delta_i \hat{\otimes} \Delta_j).$$

Since $DT = TD$ (by the uniqueness of D), it is possible to choose $\xi_{ij}^0 = D(\Delta_i \hat{\otimes} \Delta_j)$ if $i < j$ and $\xi_{ij}^0 = 0$ if $i > j$, so the only real problem is choosing ξ_{ii}^0. However, $D(\Delta_i \hat{\otimes} \Delta_i) = TD(\Delta_i \hat{\otimes} \Delta_i)$, so the existence of a suitable ξ_{ii}^0 follows at once from the fact that in dimensions greater than i the chain complex $N(X[i] \otimes X[i])$ is free over the $\mathbf{Z}/2$ group ring of $\{1, T\}$. (This is because the fixed point set of the action of T on the product $\Delta[i] \times \Delta[i]$ is i-dimensional.)

Suppose that maps D^l with the stated properties have been found for $l < k$ $(k > 0)$, in other words, that appropriate ξ_{ij}^l have been found for $l < k$. The elements ξ_{ij}^k must then satisfy

$$\xi_{ij}^k + T\xi_{ji}^k = y_{ij} \qquad (i + j \geqslant 2k, (i,j) \neq (k, k)),$$
$$\xi_{kk}^k + T\xi_{kk}^k = y_{kk} + \phi_k(\Delta_k \hat{\otimes} \Delta_k),$$

where $y_{ij} = D^{k-1}\partial(\Delta_i \hat{\otimes} \Delta_j) + \partial D^{k-1}(\Delta_i \hat{\otimes} \Delta_j)$. A computation using the inductive assumption shows that $y_{ij} + Ty_{ji} = 0$, so if $i \neq j$ or if $i = j > k$ we can proceed exactly as above in the case $k = 0$ to choose ξ_{ij}^k.

The choice of ξ_{kk}^k is more complicated, because $N(X[k] \otimes X[k])_k$ is *not* free as a module over the $\mathbf{Z}/2$-group ring of $\{1, T\}$. Let

$$\nu\colon N(X[k] \otimes X[k]) \to N(X[k] \otimes X[k])$$

be the chain map given by $\nu(y) = y + Ty$. It is easy to see that the quotient complex kernel (ν)/image (ν) is naturally isomorphic to $N(X[k])$ and that the nontrivial element in dimension k is represented by the residue class of $\phi_k(\Delta_k \hat{\otimes} \Delta_k)$. Thus to show that a suitable ξ_{kk}^k exists it is enough to show that, modulo image (ν), $y_{kk} \equiv \phi_k(\Delta_k \hat{\otimes} \Delta_k)$ or even $\partial y_{kk} \equiv \partial\phi_k(\Delta_k \hat{\otimes} \Delta_k)$. Calculating modulo image (ν) (when necessary) gives

$$\begin{aligned}\partial y_{kk} &= \partial D^{k-1}\partial(\Delta_k \hat{\otimes} \Delta_k)\\ &= \partial(D^{k-1} + D^{k-1}T)(\Delta_k \hat{\otimes} \partial\Delta_k)\\ &\equiv \partial(D^{k-1} + TD^{k-1}T)(\Delta_k \hat{\otimes} \partial\Delta_k)\\ &= \partial(\partial D^{k-2} + D^{k-2}\partial)(\Delta_k \hat{\otimes} \partial\Delta_k)\\ &= (\partial D^{k-2} + D^{k-2}\partial)(\partial\Delta_k \hat{\otimes} \partial\Delta_k)\\ &\equiv \phi_{k-1}(\partial\Delta_k \hat{\otimes} \partial\Delta_k)\\ &\equiv \partial\phi_k(\Delta_k \hat{\otimes} \Delta_k)\end{aligned}$$

which is exactly what is needed.

The construction of the maps E^k goes directly along the above lines and is left to the reader.

4. Construction of the operations. Suppose that V is a simplicial $\mathbf{Z}/2$-module and that $\mathrm{Sym}_2 V = V \otimes V/\{x \otimes y + y \otimes x\}$ is the second symmetric power of V. The purpose of this section is to define natural maps

$$\gamma_i\colon \pi_n V \to \pi_{n+1}\mathrm{Sym}_2 V, \qquad 2 \leqslant i \leqslant n.$$

If V happens to be a simplicial commutative $\mathbf{Z}/2$-algebra, multiplication gives a map $\mu\colon \mathrm{Sym}_2 V \to V$, so that operations

$$\delta_i\colon \pi_n V \to \pi_{n+i} V, \qquad 2 \leqslant i \leqslant n,$$

can be defined by setting $\delta_i = \mu_* \circ \gamma_i$.

The normalized chain complex $N(\mathrm{Sym}_2 V)$ is the quotient of $N(V \otimes V)$ by the action of the switching map T; let $\rho\colon N(V \otimes V) \to N(\mathrm{Sym}_2 V)$ be the natural quotient map. If $x \in \pi_n V$, let $a \in C(V)_n$ be a cycle representing x and define $\gamma_i(x)$ to be the residue class in $\pi_{n+i}(\mathrm{Sym}_2 V)$ of $\rho D^{n-i}(a \hat{\otimes} a)$. (The maps $\{D^k\}$ are the ones constructed in 3.3.)

4.1. LEMMA. *The element $\gamma_i(x)$ is well defined, in the sense that*

(i) *$\rho D^{n-i}(a \hat{\otimes} a)$ is a cycle in $N(\mathrm{Sym}_2 V)_{n+i}$ whose homology class does not depend on the particular cycle representative a chosen for x, and*

(ii) *if $\{\Delta^k\}$ is another sequence of maps as in* 3.2 *then the cycle $\rho\Delta^{n-i}(a \hat{\otimes} a)$ is homologous to $\rho D^{n-i}(a \hat{\otimes} a)$.*

PROOF. This follows immediately from 3.3 and the identity $\rho T = \rho$.

If x and y are elements of $\pi_n V$, let $x * y \in \pi_{2n}(\mathrm{Sym}_2 V)$ be the image under ρ_* of the exterior product $x \otimes y \in \pi_{2n}(V \otimes V)$.

4.2. LEMMA. *Suppose that x and y belong to $\pi_n V$. Then*

$$\gamma_i(x + y) = \gamma_i x + \gamma_i y \quad \text{for } 2 \leqslant i < n, \text{ and}$$
$$\gamma_n(x + y) = \gamma_n(x) + \gamma_n(y) + x * y.$$

PROOF. Let a and b be cycles in $C(V)$ representing x and y respectively. Then it follows from the identities in 3.3 that if $2 \leqslant i \leqslant n$ there is an equality

$$\rho D^{n-i}((a+b)\hat{\otimes}(a+b))$$
$$= \rho D^{n-i}(a\hat{\otimes} a) + \rho D^{n-i}(b\hat{\otimes} b) + \partial\rho D^{n-i-1}(a\hat{\otimes} b).$$

On the other hand, for $i = n$,

$$\rho D^0((a+b)\hat{\otimes}(a+b)) = \rho D^0(a\hat{\otimes} a) + \rho D^0(b\hat{\otimes} b) + \rho D(a\hat{\otimes} b).$$

Since $D(a\hat{\otimes} b)$ represents (by definition) the exterior product $x\otimes y$, this gives the desired formula.

This completes the construction of the operations γ_i and hence of the operations δ_i. It is easy to see that 4.2 implies part (i) of Theorem 2.1.

4.3. Remark. It is not possible to use the above formulas to define an operation $\delta_1\colon \pi_n V \to \pi_{n+1} V$ ($n \geqslant 1$). The problem is that if a is a cycle representative for the element $x \in \pi_n V$ ($n \geqslant 1$), then $\rho D^{n-1}(a\hat{\otimes} a)$ is *not* a cycle in $N\,\mathrm{Sym}_2 V$; in fact, by 3.3,

$$\partial\rho D^{n-1}(a\hat{\otimes} a) = \rho\phi_n(a\hat{\otimes} a) \in N(\mathrm{Sym}_2 V)_n.$$

This calculation incidentally shows that if V is a simplicial commutative $\mathbf{Z}/2$-algebra the squaring homomorphism $SQ\colon V \to V$ induces a zero map on homotopy in dimension greater than zero, since, for x and a as above, $SQ_*(x)$ is represented by the cycle $\mu\rho\varphi_n(a\hat{\otimes} a)$.

4.4. Remark. Let $\Lambda^2 V$ be the quotient of $V\otimes V$ by the relations $a\otimes a = 0$, $a \in V$, and let $\rho'\colon N(V\otimes V) \to N(\Lambda^2 V)$ be the natural quotient map. It follows from Remark 4.3 that the formulas above can be used to define natural operations

$$\gamma_i'\colon \pi_n V \to \pi_{n+i}\Lambda^2 V, \qquad 1 \leqslant i \leqslant n.$$

If V is a simplicial $\mathbf{Z}/2$-Lie algebra, the Lie bracket gives a map $\beta\colon \Lambda^2 V \to V$, so that defining $\delta_i' = \beta_* \circ \gamma_i'$ gives operations

$$\delta_i'\colon \pi_n V \to \pi_{n+i} V, \qquad 1 \leqslant i \leqslant n.$$

These operations δ_i' can be identified with the λ_i ($i > 0$) of Priddy [**9**]; the existence in addition of a λ_0 in his case is a consequence of the fact that he deals with simplicial *restricted* Lie algebras.

5. Symmetric powers. This section sets up a correspondence between the homotopy groups of symmetric powers of simplicial $\mathbf{Z}/2$-modules and some cohomology groups of symmetric groups. What we do is more or less dual to the classical idea of relating the cohomology of symmetric products of spaces to the *homology* of symmetric groups by the Steenrod reduced power construction [**5**]. The approach below is in some ways simpler than the classical one but it is less topologically oriented.

Let $V^{\otimes n}$ denote the nth tensor power of the simplicial $\mathbf{Z}/2$-module V. If G is a subgroup of the symmetric group Σ_n, let G act on $V^{\otimes n}$ by permuting factors and let $\mathrm{Sym}_G V$ denote the quotient of $V^{\otimes n}$ by this G action.

If $G \subseteq K \subseteq \Sigma_n$ there is a natural projection map $\mathrm{Sym}_G V \to \mathrm{Sym}_K V$ as well as a transfer map $\mathrm{Sym}_K V \to \mathrm{Sym}_G V$ [**3**, p. 254]; let $q_\#$ denote the induced map $\pi_*\mathrm{Sym}_G V \to \pi_*\mathrm{Sym}_K V$ and $q^\#$ the induced map $\pi_*\mathrm{Sym}_K V \to \pi_*\mathrm{Sym}_G V$. If i: $G \to K$ denotes the inclusion and M is a Σ_n-module, there is also a restriction map

$i^*\colon H^*(K, M) \to H^*(G, M)$ as well as a cohomology transfer map $i_*\colon H^*(G, M) \to H^*(K, M)$.

5.1. PROPOSITION. *Let V be a simplicial $\mathbf{Z}/2$-module and let G be a subgroup of Σ_n. Then there are natural maps*

$$\psi_G\colon \pi_i \mathrm{Sym}_G\, V \to \bigoplus_k H^k(G; \pi_{i+k}(V^{\otimes n})) \qquad (i \geqslant 0).$$

Moreover, if $G \subseteq K \subseteq \Sigma_n$ then, in the notation above,

$$\iota_* \psi_G = \psi_K q_\# \quad \textit{and} \quad \iota^* \psi_K = \psi_G q^\#.$$

This will be proved below.

Let ψ_n denote ψ_G for $G = \Sigma_n$. A simplicial $\mathbf{Z}/2$-module V is *connected* if $\pi_0 V = 0$.

5.2. PROPOSITION. *The maps ψ_n are injective for all connected simplicial $\mathbf{Z}/2$-modules V.*

It follows that if $V = S^m$ (2.3), the map ψ_2 gives injections

$$\pi_i\, \mathrm{Sym}_2\, S^m \to H^{2m-i}(\Sigma_2; \mathbf{Z}/2).$$

Let $w^k \in H^k(\Sigma_2; \mathbf{Z}/2)$ be the generator.

5.3. PROPOSITION. *If u is the generator of $\pi_m S^m$ then*

$$\psi_2(\gamma_i(u)) = w^{m-i} \qquad (2 \leqslant i \leqslant m).$$

Let G be the subgroup of Σ_4 generated by the cycles $\{(1, 2), (3, 4), (1, 3)(2, 4)\}$. Since $\mathrm{Sym}_G\, V$ is naturally isomorphic to $\mathrm{Sym}_2\, \mathrm{Sym}_2\, V$, Lemma 5.1 provides natural maps

$$\psi_G\colon \pi_i\, \mathrm{Sym}_2\, \mathrm{Sym}_2\, S^m \to H^{2m-i}(G; \mathbf{Z}/2).$$

It is known that the Serre spectral sequence of the group extension

$$1 \to \langle (1, 2), (3, 4) \rangle \to G \to \Sigma_2 \to 1$$

canonically collapses [**1**, §13] and gives an isomorphism

$$H^*(G; \mathbf{Z}/2) \approx H^*(\Sigma_2; H^*(\Sigma_2; \mathbf{Z}/2) \hat{\otimes} H^*(\Sigma_2; \mathbf{Z}/2))$$

where the outer Σ_2 on the right acts on its graded coefficient module by the switching map. In particular, each Σ_2-invariant element $w^i \hat{\otimes} w^i$ gives rise to classes $w^k \otimes w^i \hat{\otimes} w^i \in H^{2i+k}(G; \mathbf{Z}/2)$.

5.4. PROPOSITION. *If u is the generator of $\pi_m S^m$ and G is as above, then*

$$\psi_G(\gamma_j \gamma_i(u)) = w^{m+i-j} \otimes w^{m-i} \hat{\otimes} w^{m-i} \qquad (2 \leqslant i \leqslant m, 2 \leqslant j \leqslant m + i).$$

The rest of this section is given over to proofs. In proving 5.1 and 5.2 we will always work with *finite dimensional* simplicial $\mathbf{Z}/2$-modules V, that is, V such that $(NV)_i = 0$ for sufficiently large i. The general case can be handled by a limit argument.

5.5. TOTAL RIGHT DERIVED FUNCTORS. Suppose that G is a group and that C is a chain complex of $\mathbf{Z}/2\ [G]$-modules which is *bounded above*, that is, $C_i = 0$ for sufficiently large i. An *injective resolution* of C is a bounded above chain complex I of injective $\mathbf{Z}/2\ [G]$-modules together with a map $C \to I$ that induces isomorphisms $H_i(C) \approx H_i(I)$, $\forall i$. (Note that C and I are allowed to extend infinitely in the negative direction.)

If F is an additive functor from the category of $\mathbf{Z}/2\ [G]$-modules to some other abelian category, let $\mathbf{R}F(C)$ denote the chain complex $F(I)$, where I is some injective resolution of C. Standard homological algebra shows that $\mathbf{R}F(C)$ is well defined up to chain homotopy type, and that there is a natural chain homotopy class of maps $F(C) \to \mathbf{R}F(C)$. The chain complex $\mathbf{R}F(C)$ is called the *total right derived functor* of F applied to C [**10**, I, 4.2].

The following properties of $\mathbf{R}F(\cdot)$ are well known.

(5.6) *There is a natural chain homotopy equivalence*

$$\mathbf{R}F(C_1 \oplus C_2) \sim \mathbf{R}F(C_1) \oplus \mathbf{R}F(C_2).$$

(5.7) *If $f\colon C_1 \to C_2$ induces an isomorphism $H_*(C_1) \approx H_*(C_2)$, then f induces a chain homotopy equivalence $\mathbf{R}F(C_1) \approx \mathbf{R}F(C_2)$.*

(5.8) *Suppose that F is an additive functor taking $\mathbf{Z}/2\ [G]$-modules to $\mathbf{Z}/2\ [G']$-modules, and that F' is an additive functor from $\mathbf{Z}/2\ [G']$-modules to some abelian category. Then, if F takes injective $\mathbf{Z}/2\ [G]$-modules to injective $\mathbf{Z}/2\ [G']$-modules, there is a natural chain homotopy equivalence*

$$\mathbf{R}(F' \circ F)(C) \sim \mathbf{R}F'(\mathbf{R}F(C)).$$

Let $\mathbf{R}^iF(C)$ denote $H_{-i}\mathbf{R}F(C)$. The groups $\mathbf{R}^iF(C)$ depend functorially on C, and there are natural maps $H_iF(C) \to \mathbf{R}^{-i}F(C)$.

(5.9) *If C_i is an injective $\mathbf{Z}/2\ [G]$-module for $i > n$, then the map $H_iF(C) \to \mathbf{R}^{-i}F(C)$ is an isomorphism for $i > n+1$ and an injection for $i = n+1$.*

(5.10) *If the chain complex C has a trivial differential, then $\mathbf{R}^iF(C)$ is naturally isomorphic to the sum $\bigoplus_k F^k(C_{k-i})$, where F^k is the kth right derived functor of F.*

PROOF OF 5.1. Suppose that G is a *finite* group and that F is the functor $H_0(G; -)$. In this case multiplication by the norm element $\Sigma_{g\in G}g$ in $\mathbf{Z}/2\ [G]$ gives a natural transformation $F(-) \dot{\to} H^0(G; -)$ which is an isomorphism for all free, and hence for all injective, $\mathbf{Z}/2\ [G]$-modules [**4**, p. 420]. It follows easily that the ith right derived functor F^i of F is naturally isomorphic to $H^i(G; -)$.

If V is a finite dimensional simplicial $\mathbf{Z}/2$-module and G is a subgroup of Σ_n, then the considerations of 5.5 give maps

$$\pi_i \operatorname{Sym}_G V = H_iF(N(V^{\otimes k})) \to \mathbf{R}^{-i}F(N(V^{\otimes k})).$$

However, by the Eilenberg-Zilber theorem (3.1)–(3.2) there is a G-equivariant homology equivalence

$$N(V) \hat{\otimes} \cdots \hat{\otimes} N(V) \to N(V^{\otimes k}) \qquad (k \text{ factors})$$

and, since $\mathbf{Z}/2$ is a field, $N(V)$ is canonically chain homotopy equivalent to a chain complex W with trivial differential. The calculation of $\mathbf{R}^{-i}F(N(V^{\otimes k}))$ follows from (5.7) and (5.10). It is easy to see that the indicated equations involving the transfer hold by looking at the way the norm element enters into the identification of $F^i(-)$ with $H^i(G; -)$.

The proof of 5.2 will follow a sequence of lemmas. Call a subgroup G of Σ_n *monic* if the map ψ_G is an injection for every connected finite-dimensional simplicial $\mathbf{Z}/2$-module V.

5.11. LEMMA. *The group* Σ_2 *is monic.*

PROOF. Let $V = S^n$ $(n > 0)$. Then $\pi_i \operatorname{Sym}_2 V = 0$ for $i < n$, because $(\operatorname{Sym}_2 V)_i = 0$ for $i < n$, and $\pi_n \operatorname{Sym}_2 V = 0$ by (4.3), because $(N \operatorname{Sym}_2 V)_n$ is a single copy of $\mathbf{Z}/2$ generated by the cycle $\phi_n(a \otimes a)$, where $a \in (S^n)_n$ is the generator. The injectivity of ψ_2 for $V = S^n$ then follows from 5.9, since $N(S^n \otimes S^n)_i$ is a free $\mathbf{Z}/2[\Sigma_2]$-module for $i > n$.

An arbitrary connected V is simplicially homotopy equivalent to a direct sum of S^{n_i}'s for various n_i, so by a limit argument it suffices to show that if ψ_2 is injective for V and for W, it is also injective for $V \oplus W$. This follows easily from (5.6), (5.9) and the fact that as a Σ_2 chain complex $N((V \oplus W)^{\otimes 2})$ is isomorphic to the sum

$$N(V \otimes V) \oplus N(W \otimes W) \oplus N[(V \otimes W) \oplus (W \otimes V)].$$

If $G \subseteq \Sigma_n$ and $K \subseteq \Sigma_m$, let $G \times K$ denote the obvious product subgroup of Σ_{n+m}.

5.12. LEMMA. *If G and K are monic, so is $G \times K$.*

PROOF. Let $F_1 = H_0(G; -)$, $F_2 = H_0(K; -)$, $F = H_0(G \times K; -)$. It follows easily from the Eilenberg-Zilber theorem that the homology map induced by

$$F(N(V^{\otimes(n+m)})) \to \mathbf{R}F(N(V^{\otimes(n+m)}))$$

can be identified with the homology map induced by the tensor product homomorphism

$$F_1(N(V^{\otimes n})) \hat{\otimes} F_2(N(V^{\otimes m})) \to \mathbf{R}F_1(N(V^{\otimes n})) \hat{\otimes} \mathbf{R}F_2(N(V^{\otimes m})).$$

The lemma follows from the Künneth formula.

Suppose that $G \subseteq \Sigma_n$ and $K \subseteq \Sigma_m$. Then there is an isomorphic copy of G in Σ_{nm} which normalizes $K^n = K \times \cdots \times K$ (n factors) and acts via conjugation on K^n by permuting the factors in the obvious way. The subgroup of Σ_{nm} generated by K^n and this isomorphic copy of G is the *wreath product* G wr K [**6**, p. 92].

5.13. LEMMA. *If G and K are monic, so is G* wr *K.*

PROOF. There is a short exact sequence

$$1 \to K^n \to G \text{ wr } K \to G \to 1$$

of groups. Let $F_1 = H_0(G; -)$ and $F_2 = H_0(K^n; -)$, where F_2 is considered as a functor from $\mathbf{Z}/2\,[G \text{ wr } K]$-modules to $\mathbf{Z}/2\,[G]$-modules. The functor F_2 preserves injectives so, if $F = H_0(G \text{ wr } K; -)$, $\mathbf{R}F = \mathbf{R}F_1 \circ \mathbf{R}F_2$ (5.8). Under this identification, the map $FN(V^{\otimes nm}) \to \mathbf{R}F(N(V^{\otimes nm}))$ factors as a composite

$$F_1F_2(C) \xrightarrow{\alpha} \mathbf{R}F_1(F_2(C)) \xrightarrow{\beta} \mathbf{R}F_1(\mathbf{R}F_2(C))$$

where $C = N(V^{\otimes nm})$. The homology map α_* is injective because it is exactly the map ψ_G for the simplicial $\mathbf{Z}/2$-module $\mathrm{Sym}_K V$ (note that $F_2(C) = N((\mathrm{Sym}_K V)^{\otimes n}))$. Let $F_3 = H_0(K; -)$. By the Eilenberg-Zilber theorem (3.1)–(3.2) the map $F_2(C) \to \mathbf{R}F_2(C)$ is G-equivariantly homology equivalent to the n-fold $\hat{\otimes}$ power of the natural map

$$\tilde{\psi}_K: F_2(N(V^{\otimes m})) \to \mathbf{R}F_2(N(V^{\otimes m})).$$

The homology map ψ_K induced by $\tilde{\psi}_K$ is injective and therefore, since $\mathbf{Z}/2$ is a field, $\tilde{\psi}_K$ has a left inverse up to chain homotopy. It follows that the n-fold $\hat{\otimes}$ power of $\tilde{\psi}_K$ has a left inverse up to G-equivariant chain homotopy and consequently, by 5.7, that β_* has a left inverse. This completes the proof.

5.14. LEMMA. *Suppose that K is a subgroup of Σ_n and that G is a 2-Sylow subgroup of K. Then if G is monic, so is K.*

PROOF. Since the index of G in K is odd, the compositions $q_{\#}q^{\#}$ and i_*i^* (in the notation of 5.1) are both isomorphisms [**3**, p. 255]. This implies that ψ_K is a retract of ψ_G.

PROOF OF 5.2. This follows from the previous lemmas and the fact that, as a permutation group, the 2-Sylow subgroup of Σ_n is isomorphic to a product of iterated wreath products of Σ_2 [**6**, p. 92].

PROOF OF 5.3. Let T denote the nontrivial element of Σ_2. Define a Σ_2 chain complex C by

$$C_i = \begin{cases} \text{free } \mathbf{Z}/2\,[\Sigma_2] \text{ module generated by } x_i & (n+1 \leqslant i \leqslant 2n), \\ \text{trivial } \Sigma_2 \text{ module } \mathbf{Z}/2 \text{ generated by } y & (i = n), \\ 0 & \text{otherwise,} \end{cases}$$

where $\partial(x_i) = x_{i-1} + Tx_{i-1}$ $(i > n+1)$ and $\partial x_{n+1} = y$. Let $a \in (S^n)_n$ denote the unique nonzero element. By 3.3 there is a Σ_2-equivariant homology equivalence f: $C \to N(S^n \otimes S^n)$ given by $f(x_i) = D^{2n-i}(a \hat{\otimes} a)$ $(n+1 \leqslant i \leqslant 2n)$, $f(y) = \phi_n(a \hat{\otimes} a)$. It follows from 5.7 and 5.9 that f induces isomorphisms

$$H_i(F(C)) \approx H_iFN(S^n \otimes S^n) = \pi_i \,\mathrm{Sym}_2 S^n \approx \mathbf{R}^{-i}F(C) \qquad (n+2 \leqslant i \leqslant 2n)$$

where $F = H_0(\Sigma_2; -)$. The proposition follows from the fact that for $n+2 \leqslant i \leqslant 2n$ the residue class of x_i in $F(C)$ is a cycle whose image in $FN(S^n \otimes S^n)$ represents, by definition, $\gamma_i(u)$.

PROOF OF 5.4. Note that G is the wreath product Σ_2 wr Σ_2. The result follows from 5.3 and a naturality argument using the factorization of ψ_G constructed in the proof of 5.13.

6. Calculations with the transfer. In this section parts (ii) and (iii) of 2.1 will be proved by making calculations with the transfer map in the cohomology of symmetric groups. This is analogous to the way in which the Adem relations can be derived by studying inclusion-induced maps in the homology of symmetric groups [**1**].

PROOF OF 2.1(ii). Suppose $m = 0$ and let $a \in V_0 = C(V)_0$ be a cycle representing y. Multiplication by a and its degeneracies gives a simplicial map $m(a)$: $V \to V$, and it is easy to see that the diagram

$$\begin{array}{ccc} \mathrm{Sym}_2\, V & \overset{\mathrm{Sym}_2(m(a))}{\rightarrow} & \mathrm{Sym}_2\, V \\ \downarrow \mu & & \downarrow \mu \\ V & \overset{m(a^2)}{\rightarrow} & V \end{array}$$

commutes, where the vertical maps are induced by the multiplication on V. The desired result follows from the naturality of the operations γ_i (§4). The case $n = 0$ is identical.

Suppose $n > 0$, $m > 0$. Choose maps f: $S^n \to V$ and g: $S^m \to V$ representing x and y respectively (2.3). Then xy is represented by a product map h: $S^n \otimes S^m \to V$. Let $z \in \pi_{n+m}(S^n \otimes S^m)$ be the generator. By naturality $\delta_k(xy)$ is the image of $\gamma_k(z) \in \pi_{n+m+k}(\mathrm{Sym}_2(S^n \otimes S^m))$ under the composite

$$\mathrm{Sym}_2(S^n \otimes S^m) \overset{\mathrm{Sym}_2\, h}{\rightarrow} \mathrm{Sym}_2\, V \overset{\mu}{\rightarrow} V.$$

This composite factors through the projection $\mathrm{Sym}_2(S^n \otimes S^m) \to (\mathrm{Sym}_2\, S^n) \otimes \mathrm{Sym}_2(S^m)$. By 5.1, 5.2 and an Eilenberg-Zilber argument there is a commutative diagram

$$\begin{array}{ccc} \pi_{n+m+k}(\mathrm{Sym}_2(S^n \otimes S^m)) & \rightarrow & H^{n+m-k}(\Sigma_2; \mathbf{Z}/2) \\ \downarrow & & \downarrow \\ \pi_{n+m+k}(\mathrm{Sym}_2\, S^n \otimes \mathrm{Sym}_2\, S^m) & \rightarrow & H^{n+m-k}(\Sigma_2 \times \Sigma_2; \mathbf{Z}/2) \end{array}$$

where the horizontal maps are injections and the right vertical map is the transfer associated to the inclusion of the diagonal Σ_2 in the product $\Sigma_2 \times \Sigma_2$. Since the corresponding restriction map $H^*(\Sigma_2 \times \Sigma_2; \mathbf{Z}/2) \to H^*(\Sigma_2; \mathbf{Z}/2)$ is surjective, it follows easily [**3**, p. 255, (6)] that this transfer is zero. This completes the proof.

PROOF OF 2.1(iii). Assume without loss of generality that x is the generator of π_n Sym S^n (2.3). By naturality, $\delta_j\delta_i(x)$ is the image of the element $\gamma_j\gamma_i(x) \in \pi_{n+i+j}$ $\mathrm{Sym}_2\, \mathrm{Sym}_2\, S^n$ under the obvious map $\mathrm{Sym}_2\, \mathrm{Sym}_2\, S^n \to \mathrm{Sym}_4\, S^n \subseteq \mathrm{Sym}\, S^n$. By 5.1, 5.2 there is a commutative diagram

$$\begin{array}{ccc} \pi_{n+i+j}\, \mathrm{Sym}_2\, \mathrm{Sym}_2\, S^n & \overset{\psi_G}{\rightarrow} & H^{3n-i-j}(G; \mathbf{Z}/2) \\ \downarrow & & \downarrow i_* \\ \pi_{n+i+j}\, \mathrm{Sym}_4\, S^n & \overset{\psi_4}{\rightarrow} & H^{3n-i-j}(\Sigma_4; \mathbf{Z}/2) \end{array}$$

where $G = \Sigma_2$ wr Σ_2 is the standard 2-Sylow subgroup of Σ_4 (cf. 5.4), the map i_* is the transfer, and the lower horizontal map is injective. By 5.4, this reduces the problem to showing that the element

$$\sum_{(j+1)/2 < k < i} \binom{i-j+k-1}{i-k} w^{n-i-j+2k} \otimes w^{n-k} \hat{\otimes} w^{n-k}$$

of $H^{3n-i-j}(G; \mathbf{Z}/2)$ is in the kernel of i_*. Writing $m = 2i - j$, $l = i - k$, $p = n - i$ puts this element in the more convenient form

$$\sum_{0 \leq l \leq (m-1)/2} \binom{m-l-1}{l} w^{p+m-2l} \otimes w^{p+l} \hat{\otimes} w^{p+l}. \tag{6.1}$$

In principle the fact that this is in the kernel of i_* could be proved by dualizing the homology transfer formulas of [**6**], but it seems a lot easier to take a direct approach.

If π and σ are groups, where σ is a subgroup of π, we will write $r[\pi, \sigma]$ for the restriction map $H^*(\pi) \to H^*(\sigma)$ and $\tau[\sigma, \pi]$ for the transfer map $H^*(\sigma) \to H^*(\pi)$ (all unspecified cohomology is with untwisted $\mathbf{Z}/2$ coefficients). Recall that, as a subgroup of Σ_4, G is generated by the cycle set $\{(1, 2), (3, 4), (1, 3)(2, 4)\}$; let $A \subseteq G$ be the subgroup generated by $\{(1, 2)(3, 4), (1, 3)(2, 4)\}$ and $B \subseteq G$ the subgroup generated by $\{(1, 2), (3, 4)\}$. Both A and B are abstractly isomorphic to the product $\Sigma_2 \times \Sigma_2$ so that $A \approx H_1(A)$, $B \approx H_1(B)$. Let $\alpha_1, \alpha_2 \in H^1(A)$ and $\beta_1, \beta_2 \in H^1(B)$ be cohomology classes dual to the given generators of A and B. Note that a basis for $H^n(A)$ $(n > 0)$ is given by the cup products $\alpha_1^i \alpha_2^j$ $(i + j = n)$; there is a similar basis for $H^n(B)$.

For reasons of clarity, from now on we will write $w^i \otimes \beta_1^j \beta_2^j$ instead of $w^i \otimes w^j \hat{\otimes} w^j$. Let $\beta(i, j)$ $(i < j)$ denote the element $\tau[B, G](\beta_1^i \beta_2^j)$ of $H^{i+j}(G)$.

LEMMA 6.2. *A $\mathbf{Z}/2$-basis for $H^*(G)$ is given by $\beta(i, j)$ $(0 \leq i < j)$ and $w^i \otimes \beta_1^j \beta_2^j$ $(0 \leq i, 0 < j)$. These elements have the following properties upon restriction to $H^*(A)$ or $H^*(B)$:*

$$r[G, B](\beta(i, j)) = \beta_1^i \beta_2^j + \beta_1^j \beta_2^i,$$

$$r[G, B](w^i \otimes \beta_1^j \beta_2^j) = \begin{cases} \beta_1^j \beta_2^j, & i = 0, \\ 0, & i > 0, \end{cases}$$

$$r[G, A](\beta(i, j)) = 0,$$

$$r[G, A](w^i \otimes \beta_1^j \beta_2^j) = \alpha_2^i \alpha_1^j (\alpha_1 + \alpha_2)^j.$$

PROOF. The statement about the given elements forming a basis for $H^*(G)$ is implicit in the remarks preceding Lemma 5.4; it is proved, in a dual homology form, in [**1**, §13]. The first and third equalities in the lemma follow from the double coset formula [**3**, p. 257]; the second is obvious; and the fourth is a restatement of the main calculation of [**1**, §21].

LEMMA 6.3. *The following formulas hold for the transfer map $\tau[A, G]$: $H^*(A) \to H^*(G)$:*

$$\tau[A, G](\alpha_1^m) = \sum_{0 \leq l \leq (m-1)/2} \binom{m-l-1}{l} w^{m-2l} \otimes \beta_1^l \beta_2^l,$$

$$\tau[A, G](\alpha_2^m) = 0.$$

PROOF. The second equality follows from [**3**, p. 255] and the fact that α_2^m is in the image of the restriction map $r[G, A]$. For the first equality, note that by the double coset formula [3, p. 257] the composite $r[G, B]\tau[A, G]$ is zero, so that (6.2)

$$\tau[A, G](\alpha_1^m) = \sum_{i,j} c_{ij}^m w^i \otimes \beta_1^j \beta_2^j.$$

Write $q_m(x, y) = \sum c_{i,j}^m y^i x^j$. Restricting $\tau[A, G](\alpha_1^m)$ to A and computing the result on the one hand with 6.2, on the other by the double coset formula, gives the equation

$$q_m(\alpha_1(\alpha_1 + \alpha_2), \alpha_2) = \alpha_1^m + (\alpha_1 + \alpha_2)^m.$$

From this it follows that

$$q_m(x, y) = yq_{m-1}(x, y) + xq_{m-2}(x, y)$$

which implies

$$\sum_m q_m(x, y)t^m = \frac{yt}{1 + t^2x + ty}.$$

The stated formula for $q_m(x, y)$ comes from expanding the right-hand side of this identity.

Let σ: $A \to A$ be the map which switches the generators, so that $\sigma^*(\alpha_1) = \alpha_2$, $\sigma^*(\alpha_2) = \alpha_1$. The automorphism σ is realized by conjugation with the cycle (2, 3) of Σ_4, so, by [3, pp. 255–256], $\tau[A, \Sigma_4] = \tau[A, \Sigma_4] \circ \sigma^*$. Since $\tau[A, \Sigma_4] = \tau[G, \Sigma_4] \circ \tau[A, G]$, it follows that for any $x \in H^*(A)$ the sum

$$\tau[A, G](x) + \tau[A, G](\sigma^*(x))$$

is an element of $H^*(G)$ which is in the kernel of the transfer $\tau[G, \Sigma_4]$.

Applying this observation to α_1^m gives that the element

$$\sum_{0 < l < (m-1)/2} \binom{m - l - 1}{l} w^{m-2l} \otimes \beta_1^l \beta_2^l \tag{6.4}$$

is in the kernel of $\tau[G, \Sigma_4]$. This is 6.1 with $p = 0$. To get the general case, note that $w^1 \otimes \beta_1 \beta_2 \in H^3(G)$ is in the image of the restriction map $r[\Sigma_4, G]$; in fact, by 6.2 and the double coset formula,

$$w^1 \otimes \beta_1 \beta_2 = r[\Sigma_4, G]\tau[G, \Sigma_4](w^1 \otimes \beta_1 \beta_2).$$

It follows [3, p. 256] that the kernel of $\tau[G, \Sigma_4]$ is closed under cup product with $w^1 \otimes \beta_1 \beta_2$. The argument is completed by noting that

$$(w^1 \otimes \beta_1 \beta_2)(w^i \otimes \beta_1^i \beta_2^j) = w^{i+1} \otimes \beta_1^{j+1} \beta_2^{j+1},$$

so that 6.1 can be obtained by multiplying 6.4 with $(w^1 \otimes \beta_1 \beta_2)^p$.

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